Synchronization in adaptive weighted networks

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In this paper, global synchronization in coupled oscillator networks is investigated. We propose an adaptive weighted network and show that such a simple and quite general scheme is able to tip oscillator networks towards collective synchronization. In comparison with the results based on linear stability analysis of unweighted networks, the proposed scheme improves the synchronizability of network dynamics, and is beneficial to analyze the effect of network structure on synchronizability.

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I. INTRODUCTION

Recently, an interest highly focuses on dynamics in networks of coupled (chaotic) oscillators, particularly when individual oscillators behave in union. Such collective behavior of networks, called collective synchronization, simulates a remarkable spontaneous phenomenon of self-organizing complex systems, which has been observed extensively in nature, ranging from living oscillators to nonliving oscillators at every scale from the nucleus to the cosmos [1–4]. A theoretic question of this subject is: How to design a coupling scheme to generate the synchronous behavior, and how to determine the stability of synchronous motion in regard to the coupling strength? Another central question is: How does connectivity topology in networks affect synchronizability? Over the last years these questions have been intensively investigated in both limit-cycle and chaotic oscillator networks.

Some effective approaches based on linear stability analysis have been proposed to investigate the stability of collective synchronization in regard to the fixed coupling strength in unweighted networks [5–10]. Particularly, in Refs. [7–10] a master stability function based on the largest Lyapunov exponent was given to calculate the linear stability of a solution in the synchronous manifold. Some interesting results were obtained with this method. For examples, collective synchronization may be lost by increasing the coupling, which is contrary to intuition; particular coupling schemes in chaotic oscillator networks may have an upper limit on the number of coupled oscillators due to short-wavelength bifurcation, over which synchronous state cannot be stable no matter how the coupling strength is adjusted. Furthermore, this idea was used to investigate how the network structure (small-world and scale-free networks) affects synchronizability, and it was found that the heterogeneity of networks would decrease synchronizability [11,12], which is opposite to the nature of dynamics in small-world networks.

Note that in the methods based on eigenvalue analysis the calculation of Lyapunov exponents is used to detect linear stability of synchronous state, so such synchronization is locally stable and applicable for the case that several attractors coexist. However, when each individual oscillator has the near-nonhyperbolicity, e.g., neuron model possessing simultaneously fast and slow variables, the approximate linearity based on the variational equations is probably problematic to deal with such nonlinear systems. In addition, for the complex coupling networks or those networks with time-dependent coupling, the calculations of eigenvalues of connection matrix and Lyapunov exponents are difficult. To avoid the linear stability analysis based on eigenvalues' calculation, the nonlinear stability of synchronous state (i.e., a stronger global stability) should be considered. Global synchronization implies that solutions from arbitrary initial values converge to the synchronous manifold, so the needed scheme is much highly required and it is crucial to determine the bound of coupling strength. Considering the spontaneity of self-organizing complex systems, one will image naturally that the needed coupling strength is chosen adaptively (not artificially fixed as in the literature) in the course of achieving collective synchronization. Namely, the coupling maybe is time dependent.

Therefore, based on the adaption idea proposed by the authors in Ref. [13–16], here we give a simple adaptive coupling to explore the collective synchronization in weighted networks. Such adaptive weighted scheme shows that the coupling strengths needed for global synchronization can be attained through a simple adaptive law. Moreover, the proposed scheme improves the synchronizability of network dynamics, e.g., the upper limit of the number of nodes, which was found to constrain the synchronization of unweighted networks, may be avoided. In addition, the proposed adaptive weighted networks are beneficial to analyze the effect of network structure on synchronizability because it is not required to calculate additive parameters, e.g., eigenvalues of connectivity matrix (i.e., Laplacian matrix).

II. SETUP OF ADAPTIVE WEIGHTED NETWORKS

In coincidence with most of the treatments in the literature, our analysis will be limited to a network of oscillators that are all strictly identical. Suppose that the dynamical behavior of each oscillator is governed by an m-dimensional ordinary differential equation

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$$\dot{X} = F(X), \quad X = (x_1, x_2, \dots, x_m) \in R^m, \tag{1}$$

where $F(X)$ is a differentiable nonlinear vector function. And we assume that $F(X)$ satisfies a very loose condition, i.e., the *uniform Lipschitz condition* with constant l defined in Refs. [13–16]. We consider networks of such N nodes, dynamics of which is governed by (mN)-dimensional equations:

$$\dot{X}^i = F(X^i) + \sum_{j=1}^{N} K_{i,j} \otimes [H(X^j) - H(X^i)], \quad i = 1, 2, \dots, N, \tag{2}$$

where $K_{i,j}=(k_{i,j}^1, k_{i,j}^2, \dots, k_{i,j}^m)$ represent coupling strengths, $H: R^m \to R^m$ is an arbitrary function of each node's variables that are used as coupling signals, and the symbol $\otimes$ is defined by $(x_1, x_2, \dots, x_m) \otimes (y_1, y_2, \dots, y_m) = (x_1 y_1, x_2 y_2, \dots, x_m y_m)$. For the convenience, we always set $K_{i,i}=0$, $i=1,2,\dots,N$ and consider symmetric coupling, i.e., $K_{i,j}=K_{j,i}$. $K_{i,j}=0$ implies no coupling between the ith and jth nodes. Therefore, the choice of $K=(K_{i,j})$ also gives the connectivity topology of networks. For example, for the star coupling with the 1th node as hub, the connectivity "matrix" is

$$K = \begin{pmatrix} 0 & K_{1,2} & K_{1,3} & \cdots & K_{1,N} \\ K_{2,1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ K_{N,1} & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{3}$$

The nearest-neighbor diffusive coupling with periodic boundary conditions (i.e., ring diffusive coupling) corresponds to

$$K = \begin{pmatrix} 0 & K_{1,2} & 0 & \cdots & K_{1,N} \\ K_{2,1} & 0 & K_{2,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ K_{N,1} & 0 & \cdots & K_{N,N-1} & 0 \end{pmatrix}. \tag{4}$$

Letting $K_{N,1}=K_{1,N}=0$ in (4), it corresponds to open-ended diffusive coupling. Note that instead of a overall coupling strength used in the literature (i.e., the unweighted networks), here coupling strengths depend on not only the corresponding two nodes (i.e., $K_{i,j}$ depends on subscripts i and j), also the components of the coupled pair (i.e., $K_{i,j}$ is an m vector). Speaking simply, this implies weighted networks.

To avoid the linear stability analysis and find suitable coupling strengths for global synchronization, we let the coupling strengths in (2) be time-dependent, i.e., $K_{i,j}=K_{i,j}(t)$. Furthermore, if $K_{i,j} \neq 0$ (i.e., there is coupling between the ith and jth nodes) we let $K_{i,j}(t)$ vary adaptively according to the update law

$$\dot{K}_{i,j} = \gamma [H(X^j) - H(X^i)] \otimes [H(X^j) - H(X^i)], \tag{5}$$

where $\gamma > 0$ is an arbitrary constant, called *dissipation parameter*. In general, we choose $0 < \gamma \ll 1$ to guarantee the coupling weak. Note more generally one may let γ be a constant vector, $\gamma = \gamma_{i,j} = (\gamma_{i,j}^1, \dots, \gamma_{i,j}^m)$.

Systems (2) and (5) constitute adaptive weighted networks. Next our aim is to show the collective synchronization will be achieved under such adaptive coupling. From the geometrical viewpoint of dynamical systems, it is equivalent to prove that the synchronous manifold $M=\{X^1=X^2=\cdots=X^N\}$ is globally attractive. Note the hyperplane M is an m-dimensional invariant manifold in the system consisting of (2) and (5). Namely, arbitrary orbits starting from M stay in M forever. Instead of the linear stability analysis of each solution on this manifold, we use Lasalle invariance principle to show the whole m-dimensional manifold M is globally attractive (i.e., nonlinearly stable). For more clarity, we let H be an identity map which represents vector coupling, and all couplings be unidirectional. Now we use the star coupling connectivity (3) as an example to show the nonlinear stability of synchronous state. In this case, the system consisting of (2) and (5) is explicitly rewritten as

$$\dot{X}^1 = F(X^1), \quad \dot{X}^i = F(X^i) + K_{i,1} \otimes (X^1 - X^i), \tag{6a}$$

$$\dot{K}_{i,1} = \gamma (X^1 - X^i) \otimes (X^1 - X^i), \quad i = 2, 3, \dots, N. \tag{6b}$$

To prove m-dimensional manifold M is globally attractive in $m(2N-1)$-dimensional system (6), we introduce the scalar function

$$\frac{1}{2} \sum_{i=2}^{N} \sum_{j=1}^{m} (x_j^1 - x_j^i)^2 + \frac{1}{2\gamma} \sum_{i=2}^{N} \sum_{j=1}^{m} (k_{i,1}^j - L)^2, \tag{7}$$

where L is a suitable constant. Then applying the method similar to that in Refs. [13–16] and Lasalle invariance principle, one may prove directly the conclusion: *The bounded orbits starting from any initial values of system (6) converge to the synchronous manifold M while $K_{i,1} \to K_{i,0}$ as $t \to \infty$, where $K_{i,0}$ is a constant vector depending on the initial values.*

The result implies that the coupling strengths $K_{i,1}$ are adaptively tuned to achieve collective synchronization, and dependent on the initial values of networks dynamics. In addition, one may set the initial coupling strengths sufficiently small, say $K_{i,1}(0)=0$, to guarantee the added control $U \equiv K_{i,1} \otimes (X^1 - X^i)$ small at the beginning (although the synchronization error is perhaps big at the time). In the other side, while the coupling strengths $K_{i,1}$ increase according to the adaptive law (5) the synchronization error $(X^1 - X^i)$ will be smaller and smaller, so the control U keeps small. Namely, the present scheme may keep the added control small as possible in the course of achieving synchronization, which is significant in practice. We also note when the oscillators of a network are limit-cycle or chaotic, a particular choice of component coupling not vector coupling is adequate. In this case, the function H will be adjusted. For example, $H(x)=(x_1, 0, \dots, 0)$ corresponds to only couple the first variable of nodes, and meanwhile the coupling strength $K_{i,j}$ is naturally in the form of $(k_{i,j}^1, 0, \dots, 0)$. For typical three-dimensional chaotic systems, such as Lorenz, Rössler, and Chua systems, it is found by numerical simulations that one single component coupling is sufficient to achieve collective synchronization in such adaptive weighted networks.

We would point out that the related rigorous proof seems to be impossible (including for other networks structures).

III. SYNCHRONIZATION IN X-COUPLING RÖSSLER OSCILLATOR NETWORKS

To compare with those results based on the linear stability analysis of unweighted networks, we use the standard Rössler system

$$\dot{x} = -y - z, \quad \dot{y} = x + ay, \quad \dot{z} = b + (x - c)z \tag{8}$$

as an illustrative example. It has been shown in the literature that x component coupling between Rössler oscillators gives rise to some interesting phenomena such as short-wavelength bifurcation, so here we also choose x coupling although y component coupling is more perfect for global synchronization of Rössler oscillator networks.

Applying the adaptive scheme proposed above, we consider star networks of N Rössler oscillators with x component coupling. The network dynamics is governed by

$$\dot{x}_1 = -y_1 - z_1, \quad \dot{y}_1 = x_1 + ay_1, \quad \dot{z}_1 = b + (x_1 - c)z_1,$$

$$\dot{x}_i = -y_i - z_i + k_i(x_1 - x_i), \quad \dot{y}_i = x_i + ay_i, \quad \dot{z}_i = b + (x_i - c)z_i, \tag{9a}$$

$$\dot{k}_i = 0.01(x_1 - x_i)^2, \quad i = 2,3, \ldots ,N, \tag{9b}$$

where we have set the dissipation parameter $\gamma=0.01$. Considering the nearest-neighbor coupling networks with periodic boundary conditions (i.e., ring diffusive coupling), the corresponding model is

$$\dot{x}_i = -y_i - z_i + k_{i,i-1}(x_{i-1} - x_i),$$

$$\dot{y}_i = x_i + ay_i, \quad \dot{z}_i = b + (x_i - c)z_i, \tag{10a}$$

$$\dot{k}_{i,i-1} = 0.01(x_{i-1} - x_i)^2, \quad i = 1,2, \cdots ,N, \quad x_0 \equiv x_N. \tag{10b}$$

The dynamical model of the open-ended diffusive coupling networks is

$$\dot{x}_1 = -y_1 - z_1, \quad \dot{y}_1 = x_1 + ay_1, \quad \dot{z}_1 = b + (x_1 - c)z_1,$$

$$\dot{x}_i = -y_i - z_i + k_{i,i-1}(x_{i-1} - x_i), \quad \dot{y}_i = x_i + ay_i, \dot{z}_i = b + (x_i - c)z_i, \tag{11a}$$

$$\dot{k}_{i,i-1} = 0.01(x_{i-1} - x_i)^2, \quad i = 2, \ldots ,N. \tag{11b}$$

Here, we only give the numerical results of chaotic Rössler oscillator networks with parameters $a=b=0.2$ and $c=7$. To measure the synchronous behavior, we introduce the quantity

$$E = \sum_{i=2}^{N} (|x_i - x_1| + |y_i - y_1| + |z_i - z_1|), \tag{12}$$

which is referred to *absolute synchronization error*. In the meantime, we introduce the *average coupling strength*

$$K_1 = \frac{1}{N-1}\sum_{i=2}^{N} k_i, \tag{13}$$

$$K_2 = \frac{1}{N}\sum_{i=1}^{N} k_{i,i-1}, \tag{14}$$

and

$$K_3 = \frac{1}{N-1}\sum_{i=2}^{N} k_{i,i-1}, \tag{15}$$

respectively, to investigate the variation of the coupling strengths in (9), (10), and (11). In all three numerical experiments, the initial values may be chosen at random, and meanwhile the initial coupling strengths are set as zero. Numerical results in Figs. 1–3 show, respectively, the adaptive synchronization in chaotic Rössler oscillator networks in (9), (10), and (11) is achieved, where the number of nodes $N=40$, 8, and 40, respectively.

IV. DISCUSSION AND CONCLUSION

In comparison with the results based on linear stability analysis of unweighted networks, an interesting phenomenon is found. For star, ring diffusive and open-ended diffusive x couplings of N chaotic Rössler oscillators, the upper limit of nodes, $N_{\max}$, which was given in Ref. [7–10], is 35, 19, and 9, respectively. It implies that the network dynamics with over these numbers of nodes is impossible to synchronize no matter how the coupling strength is tuned. It is well known that the short-wavelength bifurcation results in the existence of the upper limit $N_{\max}$ in the local linear stability analysis. Actually, a similar phenomenon also arises in the case of global synchronization [17,18], e.g., due to the so-called equilibria disappearance bifurcations the global synchronization of x-coupling Rössler oscillators cannot be achieved even for the largest coupling strength, where coupling is fixed. However, as the numerical results above show, this limit size may be broken through in the present adaptive weighted networks. It shows that the proposed simple scheme improves the synchronizability of network dynamics. Here a crucial idea is that the coupling strengths vary adaptively according to the update law (5), which is different from the case of a fixed coupling strength in nature. Also note that the present networks are weighted, but the previous analysis were on unweighted networks. Just as it was recently shown that some weighted networks can enhance the synchronization of the system [19], this difference maybe contributes to the present improvement on the synchronizability of network dynamics. In addition, the direct coupling (i.e., unidirection) in these examples may be one of factors resulting in such improvement. It remains to investigate further how the present adaptive weighted coupling improves the synchronizability of network dynamics.

The proposed scheme is also quite convenient to analyze numerically the effect of network structure on synchroniz-

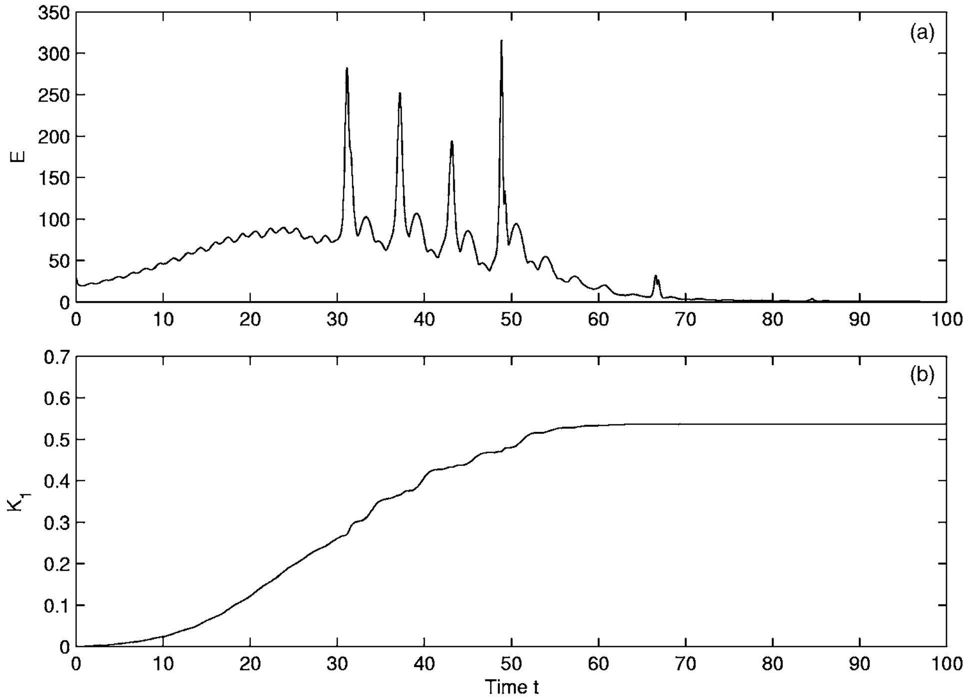

FIG. 1. The numerical results of chaotic Rössler networks with star coupling (9) where $N=40$. (a) shows the asymptotical behavior of absolute synchronization error E defined in (12), which implies the collective synchronization is achieved; (b) gives the converging course of average coupling strength K_1 defined in (13) starting from zero coupling strengths.

ability. Simply we may measure the effect of network structure by numerically checking the following two quantities: the converged average coupling strength defined as (13)–(15) and the transient (or convergence) time, i.e., the needed time for reaching synchronization. The smaller quantities imply that the corresponding network structure possesses the stronger synchronizability. We compare the effect of three network structures (i.e., star coupling, ring diffusive coupling, and open-ended diffusive coupling) on synchronizability by choice of the same initial values and number of nodes. We find the star coupling network is most easy to synchronize (i.e., the transient time is shortest and the converged average coupling strength smallest as well), and while the ring diffusive coupling is most difficult. The similar results are found in the other coupled oscillator networks. In the other side, it is well known that among these networks the star coupling is more "small-world" and more heterogeneous. Therefore the numerical results are accordant with dynamics of small-world networks, i.e., characteristic of small-world strengthens the synchronizability. And mean while it results in a difference with the findings based on the linear stability analysis of unweighted networks [11,12], where it was found that networks with a homogeneous distribution are more synchronizable than heterogeneous ones. However, this problem

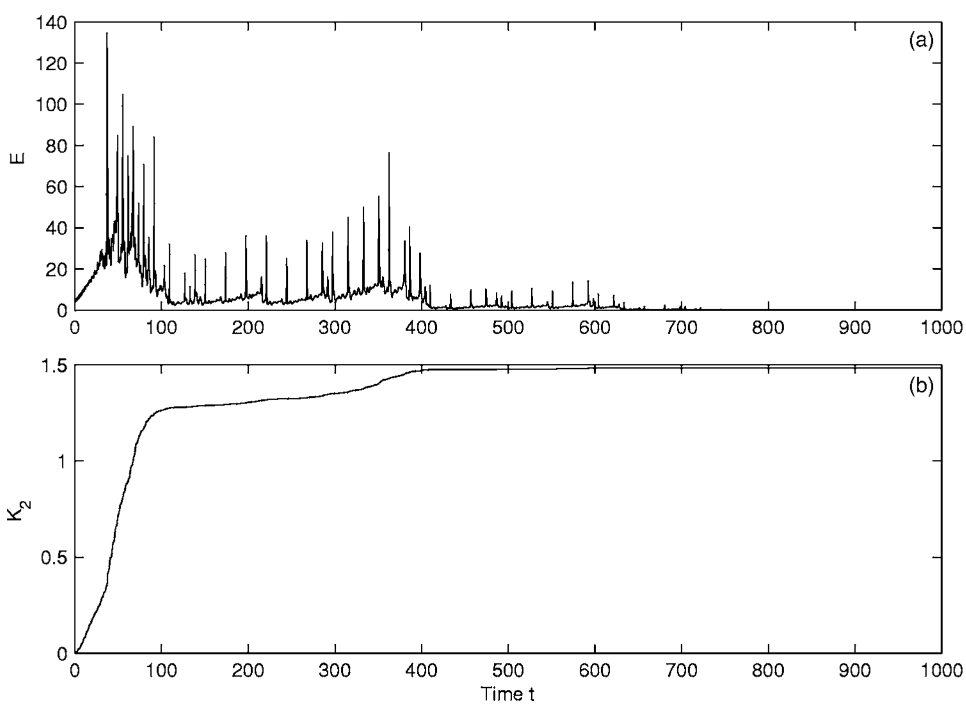

FIG. 2. (a) and (b) show numerically adaptive synchronization in chaotic Rössler networks with ring diffusive coupling (10), and variation of average coupling strength K_2 defined in (14), respectively, where $N=8$.

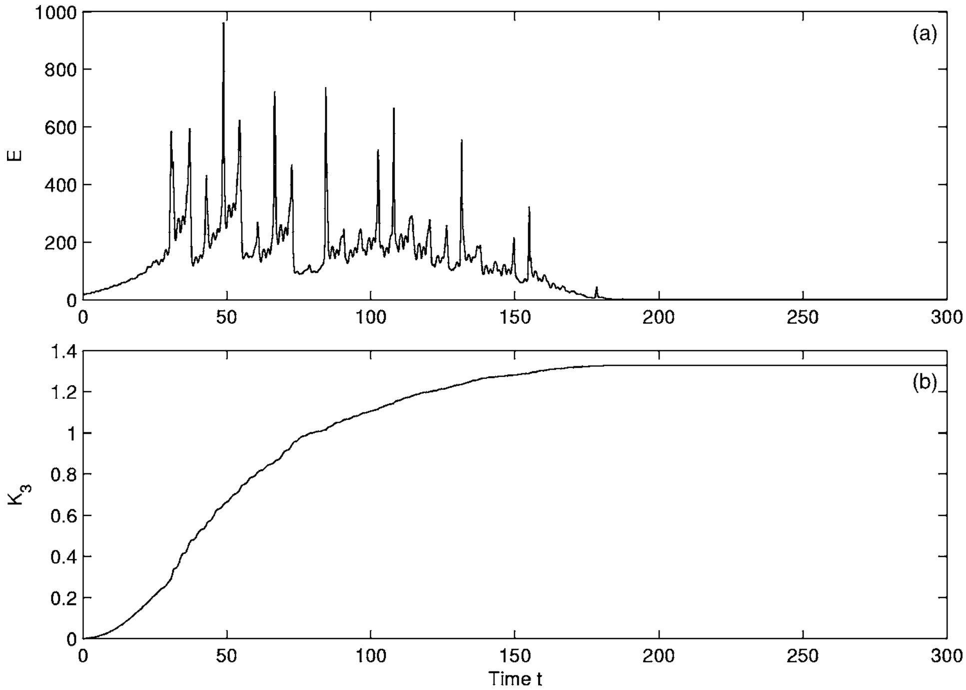

FIG. 3. (a) and (b) show numerical results of absolute synchronization error in chaotic Rössler networks with open-ended diffusive coupling (11), and average coupling strength K_3 defined in (15), respectively, where $N = 40$. Here initial values are the same as those in Fig. 1.

remains to investigate further: How does random short cut in semirandom complex networks [20,21], such as small-world and scale-free networks, affect synchronization in the present adaptive weighted networks?

In conclusion, the proposed adaptive weighted coupling gives a new insight to explore dynamics of coupled oscillator networks. In comparison with those schemes with fixed and unweighted coupling, this quite general scheme improves the synchronizability of network dynamics. And meanwhile, since it is very simple and analytical (without additional numerical calculations, e.g., matrix eigenvalues and Lyapunov exponents) the proposed scheme may be directly used to investigate the synchronization of the more complicated networks (e.g., high-dimensional structure, and even semirandom complex networks), and to analyze numerically the effect of network structure on synchronizability. In addition, theoretically the viewpoint of adaption is certainly significant with collective synchronization in biological systems, e.g., neuron synchronization [22], although presently we have no method to confirm whether just the adaption law as (5) underlies the life rhythm.

ACKNOWLEDGMENTS

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